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Some Results on the Graph Associated to a Lattice with Given a Filter

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ABSTRACT

In this paper, we study some graph-theoretical properties of $\Gamma_S(L)$, a graph which the vertex set is all elements of a finite lattice L and two distinct vertices a and b are adjacent if and only if $a \vee b \in S$, where S is a $\wedge$-closed subset of L. As a consequence of our work, some results in Afkhami and Khashayarmanesh (2014b) are extended to the case that S is a filter.

Keywords: Clique number, chromatic number, filter, complete n-partite graph, matching number.

1. Introduction

In this paper L stands for a finite bounded lattice. We start by reminding some necessary definitions of Lattice Theory from Donnellan (1968), Grätzer (2011).

An element z is maximal in $A \subset L$ if $z \leq x$, for $x \in A$, implies that $x = z$. The set of all maximal elements of the subset A of the lattice L is denoted by $Max(A)$. A nonempty subset I of a lattice L is called an ideal if it is closed under $\vee$ operation and for each element $a \in I$, $b \leq a$ imply $b \in I$. A proper ideal I of a lattice is called prime if $a \wedge b \in I$ imply $a \in I$ or $b \in I$. Dually, the concept of filter and prime filter are defined. Let $x \in L$, we set $x^{\ell} = \{y \in L : y \leq x\}$ and is said the ideal generated by x. We refer to Donnellan (1968), Grätzer (2011) for more details.

Let G be an undirected simple graph whose vertex set is $V(G)$. The complement of the graph G is denoted by $\overline{G}$. The complete graph with n vertices and the complete k -partite graph are denoted by K_n and $K_{n_1,,\dots n_k}$, respectively. A graph is called connected if there exists a path between any two distinct vertices. For connected graph G, let $d(x, y)$ be the length of shortest path between two vertices x and y. The diameter of the graph G is denoted by $diam(G)$ and is defined as $diam(G) = sup\{d(x, y) \mid x, y \in V(G)\}$.

A complete subgraph of G is called a clique of the graph G and the clique number of the graph G is the number of vertices of the largest clique in G, denoted by $\omega(G)$. The chromatic number of G is the minimal number of colors which can be color the vertices of G such that any two adjacent vertices have not same colors. The chromatic number of G is denoted by $\chi(G)$. A matching in the graph G is a set of edges without common vertices. A matching M is called perfect matching, if every vertex of G is the end of exactly one edge of M. The size of the largest independent edge set in the graph G is called the matching number of G and denoted by $m(G)$. The graph H is called an induced subgraph of G if $V(H) \subseteq V(G)$ and vertices are adjacent in H if they are adjacent in G, Bondy and Murty (1976).

The disjoint union of two graphs G and H with disjoint vertex sets is denoted by $G + H$ and is defined as follows: $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. Also $\overline{\bar{G} + \bar{H}}$ is called the join G and H and is denoted by $G \oplus H$, Hammack and Klavžar (2011).

In graph theory, the hamiltonian path is a path that visits every vertex precisely once. A hamiltonian graph is a graph in which include a hamiltonian cycle, otherwise it is called non-hamiltonian. An eulerian path in the graph G is a path that visits each edge exactely once. The graph G is called eulerian if it contains an eulerian path which starts and terminates on the same vertex. We recall two following theorems about Hamiltonian and Eulerian graph Bondy and Murty (1976).

Theorem 1.1. *The graph G is eulerian if and only if for every $v \in V(G)$, $deg(v)$ is even.*

Theorem 1.2. *If for every $v \in V(G)$, $deg(v) \geq \frac{n}{2}$, then the graph G is Hamiltonian.*

In recent years, many papers pay attention to study the graphs derived from algebraic and ordered structures, see Afkhami and Khashayarmanesh (2014a,b), Joshi (2012a,b), Visweswaran (2011). In Afkhami and Khashayarmanesh (2014b), authors introduced the notion of comaximal graph of a lattice L. In this simple graph, the set of vertices are all elements of the lattice L and two vertices a and b are adjacent if and only if $a \vee b = 1$. Also they verified most important properties of this graph. Let L be a lattice and S be a $\wedge$-closed subset of L. The graph $\Gamma_S(L)$ is defined in Afkhami and Khashayarmanesh (2014a) whose the vertex set is all elements of the lattice L and two distinct vertices a and b are adjacent if and only if $a \vee b \in S$. Malekpour and Bazigaran (2016, 2019) investigated some more properties of this graph.

In the present paper, we generalize the properties of the comaximal graph of the lattice for the graph $\Gamma_S(L)$, S is a filter, and then determine the clique number, chromatic number and matching number of this graph, for some special cases. Also the eulerian and hamiltonian property of $\Gamma_S(L)$ in some special cases were investigated.

2. Main Results

Let S be a filter of a finite bounded lattice L and consider the graph $\Gamma_S(L)$. In this section, the main properties of $\Gamma_S(L)$ is investigated, when S is a filter. Since S is an upper set, all elements of S is adjacent to all other vertices. Thus we remove all elements of S from the vertex-set and study the induced subgraph $\Gamma_S(L)$ on the vertex-set $L \setminus S$ which denote by $\Gamma^1_S(L)$.

We now consider the following two sets:

$$J'(L) := \bigcap_{m \in Max(S^c)} m^{\ell}, \ C'(L) := Max(S^c).$$

In view of (Afkhami and Khashayarmanesh, 2014b, Proposition 3.1), we conclude the following proposition:

Proposition 2.1. *The induced subgraph of $\Gamma_S(L)$ on vertex-set $J'(L)$ is totally disconnected and it is disjoint from the induced subgraph of $\Gamma_S(L)$ on $S^c \setminus J'(L)$.*

Proof. Assume that a and b are two arbitrary elements of $J'(L)$. We show that a is not adjacent to b in $\Gamma_S(L)$. Since $a, b \in J'(L)$, there exists a maximal element m in S^c such that $a, b \in m^{\ell}$. So $a \vee b \leq m$. Now if a and b are adjacent then $a \vee b \in S$. But S is an upper set, so $m \in S$, which is impossible. Also let $a \in J'(L)$ and $b \in S^c \setminus J'(L)$. There exists $m \in C'(L)$ such that $b \leq m$ and by definition of $J'(L)$, $a \leq m$, so $a \vee b \leq m$. Since S is an upper set and $m \notin S$, thus $a \vee b \notin S$, which implies that a and b are not adjacent in $\Gamma_S(L)$ and the proof is complete. □

By Proposition 2.1, in the rest of the paper, we study the induced subgraph of $\Gamma^1_S(L)$ on vertex-set $S^c \setminus J'(L)$. We denote this graph by $\Gamma^2_S(L)$. It is clear that if $|MaxS^c| = 1$ and $MaxS^c = \{m\}$, then for every element $a, b \in S^c \setminus J'(L)$, we have $a, b \leq m$. So, $a \vee b \notin S$, since S is an upper set. Thus, $|MaxS^c| = 1$ implies that the graph $\Gamma^2_S(L)$ is a totally disconnected. Also if $\Gamma^2_S(L)$ is a totally disconnected, one can easily show that $|MaxS^c| = 1$. Therefore, in the rest of this section for investigating the graph $\Gamma^2_S(L)$, we assume that $|MaxS^c| \geq 2$.

Theorem 2.1. *The graph $\Gamma^2_S(L)$ is connected and $diam(\Gamma^2_S(L)) \leq 3$.*

Proof. Consider two distinct vertices $x, y \in S^c \setminus J'(L)$. Since $x, y \notin J'(L)$, there exist two maximal elements m, m' in S^c, such that $x \notin m^{\ell}$ and $y \notin m'^{\ell}$. Thus, $x \vee m \neq m$ and $y \vee m' \neq m'$. Since, m, m' are maximal in S^c, $x \vee m > m$ and $y \vee m' > m'$, which implies that $x \vee m \notin S^c$ and $y \vee m' \notin S^c$. So $x \vee m \in S$ and $y \vee m' \in S$ which means that x is adjacent to m and y is adjacent to m'. Now if $m = m'$, then $x - m - y$ is a path between x, y. If $m \neq m'$, by maximality of m, m' in S^c, $m \vee m' \notin S^c$. So $m \vee m' \in S$, which means that m and m' are adjacent. Therefore, in this case x and y are connected by the path $x - m - m' - y$ of length 3. Hence, $diam(\Gamma^2_S(L))$ is at most 3. □

Proposition 2.2. *The graph $\Gamma_S^2(L)$ is complete if and only if $S^c \setminus J'(L) = MaxS^c$.*

Proof. Assume that the graph $\Gamma_S^2(L)$ is complete. Suppose on the contrary that $S^c \setminus J'(L) \neq MaxS^c$ and let $x \in (S^c \setminus J'(L)) \setminus MaxS^c$. Then, there exists $m \in MaxS^c$ such that $x \leq m$. So $x \vee m = m \notin S$. Hence, x and m are not adjacent in $\Gamma_S^2(L)$ which is a contraction.

Conversely, assume that $S^c \setminus J'(L) = MaxS^c$. Let $m, m' \in MaxS^c$. Since m, m' are two distinct maximal elements in S^c and $m, m' < m \vee m'$, $m \vee m' \in S$ which means that m, m' are adjacent. Hence, the graph $\Gamma_S^2(L)$ is complete. □

Corollary 2.1. *In the graph $\Gamma_S^2(L)$, we have*

$$\omega(\Gamma_S^2(L)) \geq |MaxS^c|.$$

Proof. By Proposition 2.2, the induced subgraph of $\Gamma_S^2(L)$ on vertex-set $MaxS^c$ is complete. So the result holds. □

Corollary 2.2. *In the graph $\Gamma_S(L)$, we have*

$$\omega(\Gamma_S(L)) \geq |MaxS^c| + |S|.$$

Proof. Since S is an upper set, all elements of S are adjacent with all other vertices. So by Corollary 2.1, the induced subgraph of $\Gamma_S(L)$ on vertex-set $S \cup MaxS^c$ is complete. □

Theorem 2.2. *The graph $\Gamma_S^2(L)$ is complete $n-$partite graph if and only if $|MaxS^c| = n$ and for each two elements $m, m' \in MaxS^c$, $m^\ell \cap m'^\ell = J'(L)$.*

Proof. Suppose that $\Gamma_S^2(L)$ is a complete $n-$partite graph. If $|MaxS^c| > n$, then there exists a part which contains two distinct elements $m, m' \in MaxS^c$. The maximality of m, m' in S^c ensures that $m \vee m' \in S$. So m and m' are adjacent, a contradiction to that $\Gamma_S^2(L)$ is an $n-$partite graph. Thus, $|MaxS^c| \leq n$. If $|MaxS^c| < n$, then there exists a part which doesn't have any elements in $MaxS^c$. Let x be an arbitrary vertex in this part. There exists $m \in MaxS^c$ such that $x \leq m$. So $x \vee m = m \notin S$, this implies that

x and m which are belong to different parts are not adjacent, a contraction. Therefore, $|MaxS^c| = n$. Now on the contrary suppose that there exist two distinct elements $m, m' \in MaxS^c$ such that $m^\ell \cap m'^\ell \neq J'(L)$. Assume that $x \in (m^\ell \cap m'^\ell) \setminus J'(L)$. Hence, $x \vee m = m \notin S$ and $x \vee m' = m' \notin S$, which implies that x is not adjacent to m and m'. As $\Gamma_S^2(L)$ is complete, we conclude that x, m, m' are belong to one part of the graph $\Gamma_S^2(L)$, this contradicts the maximality of m and m' in S^c.

Conversely, consider $V_i := m_i^\ell \setminus J'(L)$, where $m_i \in MaxS^c$. It is easy to check that the graph $\Gamma_S^2(L)$ is complete $n-$partite with these parts. □

In the following theorem, we characterize the clique and chromatic numbers of the graph $\Gamma_S^2(L)$, then we apply these results to obtain the same conclusion for the graph $\Gamma_S(L)$.

Theorem 2.3. *In the graph $\Gamma_S^2(L)$, we have*

$$\omega(\Gamma_S^2(L)) = \chi(\Gamma_S^2(L)) = |MaxS^c|.$$

Proof. Suppose that $|MaxS^c| = t$ and $MaxS^c = \{m_1, ..., m_t\}$. Set $S_1 = m_1^l$, $S_2 = m_2^l \setminus m_1^l$ and $S_i = m_i^l \setminus \cup_{j<i} m_j^l$. One can easily check that for all $i \neq j$, $S_i \cap S_j = \emptyset$ and $S^c = \cup_{i=1}^t S_i$. Also by definition of S_i, there is no adjacency between vertices in S_i, so $\chi(\Gamma_S^2(L)) \leq t$. By Corollary 2.1, $\omega(\Gamma_S^2(L)) \geq |MaxS^c| = t$. In view of the well known result, $\omega(\Gamma_S^2(L)) \leq \chi(\Gamma_S^2(L))$ and this completes the proof. □

Corollary 2.3. *If S is a filter of the lattice L, then,*

$$\chi(\Gamma_S(L)) = \omega(\Gamma_S(L)) = |MaxS^c| + |S|.$$

Proof. Since S is an upper set, all elements of S are adjacent together and with all other vertices. So for coloring the vertices of the graph $\Gamma_S(L)$, we should add $|S|$ colors to the chromatic number of the graph $\Gamma_S^2(L)$. By Proposition 2.1, the result holds. □

Let S be a prime filter of the lattice L, if $|MaxS^c| \geq 2$, then for each two distinct elements $m, m' \in MaxS^c$, $m \vee m' \in S$, by maximality of m, m' in S^c. Thus, $m \in S$ or $m' \in S$, which is impossible. So $|MaxS^c| = 1$.

As mentioned above and by previous corollary, we have the following theorem:

Theorem 2.4. *Let S be a prime filter of the lattice L, then*

$$\chi(\Gamma_S(L)) = \omega(\Gamma_S(L)) = |S| + 1.$$

Note that another proof of the previous theorem follows from (Malekpour and Bazigaran, 2016, Theorem 2.7).

Proposition 2.3. *Suppose that S_1 and S_2 are two filters of two lattices L, L', respectively. If $\Gamma^2_{S_2}(L_2) \cong \Gamma^2_{S_1}(L_1)$, then $|MaxS_2| = |MaxS_1|$ and $|S_1^c \setminus J'(L_1)| = |S_2^c \setminus J'(L_2)|$.*

Proof. By Theorem 2.3, the result follows. □

The sequential join $G_1 \oplus G_2 \oplus ... \oplus G_k$ of graphs G_1,...,G_k is the graph defined by $(G_1 \oplus G_2) + (G_2 \oplus G_3) + ... + (G_{k-1} \oplus G_k)$.

In the view of Theorem 2.2 and Proposition 2.1, we have next result for the graph $\Gamma_S(L)$.

Theorem 2.5. *Suppose that S is a filter of the lattice L, $|MaxS^c| = n$ and for each two distinct elements $m_i, m_j \in MaxS^c$, $m_i^\ell \cap m_j^\ell = J'(L)$. Then*

$$\Gamma_S(L) = K_{r_1,...,r_n} \oplus K_{|S|} \oplus \overline{K}_{|J'(L)|}.$$

In the following, we determine some graph-theoretic properties of the graph $\Gamma_S(L)$, where S is a prime filter or an ideal.

Theorem 2.6. *Let L be a boolean lattice and $S = \{1\}$. then $\Gamma_S(L)$ has a perfect matching. Furthere the matching number of the graph $\Gamma_S(L)$ is $\frac{|L|}{2}$.*

Proof. Consider $M = \{(x, x') \in E(\Gamma_S(L)) : x, x' \in V \setminus \{0, 1\}, x \vee x' = 1\} \cup \{(0, 1\}$. Since L is a boolean lattice, every non zero element has a unique complement. So every vertex of the graph $\Gamma_S(L)$ is incident to exactly one edge in M. Thus, the edges of M are independent and makes a perfect matching for the graph $\Gamma_S(L)$ which implies that $m(G) = \frac{|L|}{2}$. □

Theorem 2.7. *Let S be a prime filter of the lattice L. If $|S| = |S^c|$, then $\Gamma_S(L)$ has a perfect matching. Furthere $m(\Gamma_S(L)) = \frac{|L|}{2}$.*

Proof. Assume that S is a prime filter of the lattice L. By (Malekpour and Bazigaran, 2016, Theorem 2.9 and Corollary 2.10), for every $x \in S$, $deg(x) = |L| - 1$. On the other hand, by assumption $|S| = |S^c|$, so it is possible to select the distinct edges with one end in S and another end in S^c such that all of the vertices are the end vertex of exactely one edge. □

Proposition 2.4. *Assume that S is a prime filter of the lattice L. If $|L|$ is odd and $|S|$ is even, then the graph $\Gamma_S(L)$ is an eulerian graph.*

Proof. Let S be a prime filter of the lattice L. By (Malekpour and Bazigaran, 2016, Theorem 2.9 and Corollary 2.10), the graph $\Gamma_S(L)$ is semiregular. Morevere, for every $x \in S$, $deg(x) = |L| - 1$ and for every $x \in S^c$, $deg(x) = |S|$. So by Theorem 1.1, if $|L| - 1$ and $|S|$ are even, the graph $\Gamma_S(L)$ is an Eulerian graph. Hence, the proof is complete.

□

Proposition 2.5. *If S is an ideal of the lattice L, then the graph $\Gamma_S(L)$ is not Eulerian.*

Proof. By (Malekpour and Bazigaran, 2016, Theorem 2.7), $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$. So, by Theorem 1.1, the graph $\Gamma_S(L)$ is not Eulerian.

□

Proposition 2.6. *Suppose that S is a prime filter of the lattice L and $|S| \geq \frac{|L|}{2}$. Then the graph $\Gamma_S(L)$ is a hamiltonian graph.*

Proof. The result follows from (Malekpour and Bazigaran, 2016, Theorem 2.9) and Theorem 1.2. □

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